# Computational Techniques and Computational Aids in Ancient Mesopotamia

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**Abstract** Any history of mathematics that deals with Mesopotamian mathematics will mention the use of tables of reciprocals and multiplication in sexagesimal place-value notation—perhaps also of tables of squares and other higher arithmetical tables. Less likely there is a description of metrological lists and tables and of tables of technical constants. All of these belong to a complex of aids for accounting that was created during the "Ur III" period (twenty-first c. BCE). Students' exercises from the Old Babylonian period (2000–1600 BCE) teach us something about their use. First metrological lists then metrological tables were learned by heart. These allowed the translation of real measures into place-value measures in a tacitly assumed basic unit. At an advanced level, we see multiplications, where first two factors and then the product are written in sequence on a clay tablet for rough work. Problem texts show us more about the use of the metrological tables and the tables of technical constants. Neither genre allows us to see directly how additions and subtractions were made, nor how multiplications of multi-digit numbers were performed. A few errors in Old Babylonian problem texts confirm, however, that multiplications were performed on a support where partial products would disappear once they had been inserted—in a general sense, some kind of abacus. Other errors, some from Old Babylonian period and some others from the Seleucid period (third and second c. BCE), show that the "abacus" in question had four or five sexagesimal levels, and textual evidence reveals that it was called "the hand". This name was in use at least from the twenty-sixth c. BCE until c. 500 BCE. This regards addition and subtraction from early times onward, and multiplication and division in Ur III and later. A couple of problem texts from the third millennium deals with complicated divisions, namely divisions of large round numbers by 7 and by 33. They use different but related procedures, suggesting that no standard routine was at hand.

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## Introduction: The Familiar

Any history of mathematics that deals with Mesopotamian or more narrowly with Babylonian mathematics will speak of tables of reciprocals and multiplication in sexagesimal place-value notation—perhaps also of tables of squares and other higher arithmetical tables such as $n^3$ and $n^2 \times (n + 1)$.[1] It is possible though less plausible that they also mention metrological tables and tables of technical constants.

Let us start by describing this system, postponing the discussion of its use and general historical setting.

The underlying number system, as stated, was a sexagesimal place-value notation, whereas ours is a decimal place-value notation. In our notation, the digit "7" may refer to the number seven, but just as well to $7 \times 10$, $7 \times 10^2$, …, or to $7 \times 10^{-1}$, $7 \times 10^{-2}$, $7 \times 10^{-3}$, …; what is actually meant is determined by its location within the sequence of digits—its "distance from the decimal point". Similarly, a Mesopotamian digit "7" may stand for 7, $7 \times 60$, $7 \times 60^2$, …, as well as $7 \times 60^{-1}$, $7 \times 60^{-2}$, ….

In the Mesopotamian notation, however, there was no analogue of the decimal point and thus no way to determine absolute magnitude from the distance to it. There was also no sign for zero, and in principle, "16 40" might thus mean not only $(16 \times 60 + 40) \times 60^n$ but also $(16 \times 60^2 + 0 \times 60^1 + 40) \times 60^n$, etc.[2] This may

[1] See, for instance, the popularizations (Neugebauer 1934) and (Neugebauer 1957), on which many general histories build. Since they are of no particular importance in what follows, I shall not return to the higher arithmetical tables.

[2] Such intermediate zeroes only came in current use (most often not for a missing sexagesimal place but for missing units or tens) in the Seleucid epoch (third to second century BCE), even though two texts from around 1600 BCE [TMS XII and XIV, see Høyrup (2002a:15 n. 16)] indicate them occasionally, and two ambiguous fragments from the intervening period seem to suggest continuity rather than Seleucid reinvention. This is one of several indications that Mesopotamian calculators did not think of their system solely as sexagesimal but also (perhaps predominantly) as a "seximal-decimal" notation (just as Roman numerals may be thought of as "dual-quintal").

One or two lines in the extensive corpus of Seleucid astronomical texts may even contain a final zero; the interpretation is quite dubious, however [Neugebauer 1955: I. 121, 166, 208]. In any case, final zeros never came into in widespread, not to say general use.

With or without final zero, the Babylonian placeholder, a mere punctuation mark, was something quite different from our zero. Our zero, beyond serving as placeholder, is also a number, the outcome of a subtraction $a - a$. When encountering such subtractions, the Old Babylonian texts might say "one is as much as the other" or "it is missing"—or they would, literally, treat the outcome as not worth speaking about and not state any result (Høyrup 2002a: 293). The situation never occurs in later texts.

seem odd to us, but we shall see that the inherent ambiguity in this floating-point notation probably created no problems in the context where it served.

In some of the text types that we shall discuss the order of magnitude plays no role—just as it plays no role when we look at a slide ruler whether "2.5" stands for 2.5, 25 or 0.25 (indeed, the same position of the slide rule gives us $2.5 \times 4$, $25 \times 400$ and $0.25 \times 0.4$) (Fig. 1). In such cases, we may render what appears as "16 40" as 16.40 (or 16..40 if we suspect an empty intermediate order of magnitude is intended). In other texts, a specific order of magnitude is certainly meant—just as 3.1416 and certainly not 314.16 is meant where the modern slide rule writes π. If "16 40" is to be interpreted as $16 \times 60^2 + 40 \times 60$, we shall translate it 16``40`; if it is to be understood as $16 \times 60 + 40$, we shall write 16`40, and if it stands for $16 + 40 \times 60^{-1}$, we shall write 16°40′ (when it is not needed as a separator, "°" will be omitted). 30′ thus means $\frac{1}{2}$, while 10′′ means $\frac{1}{360}$.[3]

This generalization of our modern degree–minute–second notation for time and angles (which descends via ancient Greek astronomy from the Mesopotamian system) has the advantage that no zeroes are written which are not in the original text (except those indicating missing units, without which the tens could not be identified as such); one may omit the pronunciation of the 'and' and keep them as tacit knowledge, just as the Mesopotamian calculators did with their knowledge about the intended order of magnitude—*they* wrote nothing corresponding to `, ° and ′.

The place-value notation was not needed for and also hardly used for additions and subtractions; we shall return to that issue. Its purpose was to serve multiplication and division.

In our algorithm for multiplication, we make use of a multiplication table with $10 \times 10$ entries, which we learn by heart. The Mesopotamian calculators, however, did not need $60 \times 60$ entries. They were trained on tables where important "principal numbers" were multiplied by 1, 2, 3, …, 19, 20, 30, 40 and 50. So, $18 \times 37$ would have to be found as $18 \times 30 + 18 \times 7$.

The term "division" may refer either to a type of question or to a procedure. The Mesopotamian calculators were fully familiar with the question "what shall I multiply by $b$ in order to get $a$"—our equation $bq = a$, whose answer is $q = \frac{a}{b}$, but they had no *standard procedure* by which to produce directly the number $q$ from the numbers $a$ and $b$. Instead, if possible, they made use of a multiplication, finding

[3]This notation was introduced by Assyriologists in the early twentieth century. Later, various alternatives have been used, the most widespread of which will write 7`13°41′40′′ as 7,13;41,40. It is particularly advantageous in the analysis of mathematical-astronomical texts.


**Fig. 1** A circular slide rule from ca. 1960
Author's photo

$q$ as $a \times \frac{1}{b}$. For this purpose, they employed a table of reciprocals, called IGI,[4] copied so often in school that it was learned by heart—Fig. 2 shows the standard version.[5]

In most practical computation, the coarse grid provided by the standard table was sufficient. We have a few tables listing approximate reciprocals of "irregular numbers", that is, numbers that do not have a reciprocal that can be expressed as a finite sexagesimal fraction. They may have been computed as school exercises or as schoolmasters' experiments—we do not know; but in any case, they show that approximate reciprocals of irregular numbers *could* be determined. We also know a technique that was used to find the reciprocals of regular numbers that did not appear in the standard table. As a simple illustration, we may pretend that $A$ = 44.26.40 does not appear and try to find $\frac{1}{A}$. We observe that the final part of the

[4]Sumerian is conventionally transliterated as small caps (sometimes as spaced writing if we believe to know the pronunciation and as small caps or capital letters if we use sign names).

[5]Since 1.12, 1.15 and 1.20 already appear to the right as IGI 50, IGI 48 and IGI 45, respectively, they are sometimes omitted to the left; moreover, some early tables have as their first line "Of sixty, its $\frac{1}{3}$ …". Originally, the table thus seems to have been thought of as fractions of 60 and not of 1, that is, as reciprocals; since the table served in floating-point calculations, this was of no consequence.

| | | | |
|---|---|---|---|
| Of 1, its 2/3 [is] | 40 | 27, its IGI | 2 13 20 |
| Its half [is] | 30 | 30, its IGI | 2 |
| 3, its IGI | 20 | 32, its IGI | 1 52 30 |
| 4, its IGI | 15 | 36, its IGI | 1 40 |
| 5, its IGI | 12 | 40, its IGI | 1 30 |
| 6, its IGI | 10 | 45, its IGI | 1 20 |
| 8, its IGI | 7 30 | 48, its IGI | 1 15 |
| 9, its IGI | 6 40 | 50, its IGI | 1 12 |
| 10, its IGI | 6 | 54, its IGI | 1 6 40 |
| 12, its IGI | 5 | 1, its IGI | 1 |
| 15, its IGI | 4 | 1 4, its IGI | 56 15 |
| 16, its IGI | 3 45 | 1 12, its IGI | 50 |
| 18, its IGI | 3 20 | 1 15, its IGI | 48 |
| 20, its IGI | 3 | 1 20, its IGI | 45 |
| 24, its IGI | 2 30 | 1 21, its IGI | 44 26 40 |
| 25, its IGI | 2 24 | | |

**Fig. 2** Translation of the table of reciprocals

number is 6.40, which is the reciprocal of 9.[6] We therefore write $A$ as a sum, $A$ = 44.20.0 + 0.6.40, and find that $9 \times A$ = 6.39.0 + 0.1.0 = 6.40. Now, 6.40 is still the reciprocal of 9, whence $9 \times 9 \times A = 1$. Therefore, $\frac{1}{A} = 9 \times 9 = 81$.

The selection of principal numbers for multiplication tables is closely related to the table of reciprocals and its use: the only irregular number to appear as a principal number is 7, while all two-place numbers of the right column of the table of reciprocals appear, as do a few others (2.15 and 4.30) that may be derived from entries in the standard table by doubling one side and halving the other.

## Why?

In order to understand to the full how this system was used, we need to look at the purpose for which it was created. The place-value idea may have been in the air as a mere notation for centuries, but the *system* connecting notation and tables was a creation of the twenty-first century BCE,[7] a period known as "Third Dynasty of Ur" or, for simplicity, Ur III. Early in this period, an extremely centralized system of economic management was created, with overseer scribes directing labour troops and responsible for costs as well as produce. As an example, we may consider how to calculate the labour and barley values of a ditch with length $l$ and rectangular cross section $w \times d$. In practical life, horizontal extensions were measured in units NINDAN (1 NINDAN ≈ 6 m), subdivided into 12 cubits, each of which consisted of 30 fingers; vertical extensions were measured in cubits. So, firstly, $l$ and $w$ were

[6]Jöran Friberg has introduced the very adequate name "trailing part algorithm" for the technique.
[7]Here and elsewhere, I follow the "middle chronology", as do most Assyriologists.

expressed in the "basic unit" NINDAN[8] as place-value numbers, and $d$ was similarly expressed as a place-value number of cubits. To find the total volume as the product of these three numbers would now be a straightforward operation, since the unit of volume was NINDAN × NINDAN × cubit. In contrast, it would be quite laborious to find it directly, for instance, from $l$ = 8 NINDAN 3 cubit, $w$ = 2 cubit 15 fingers, $d$ = 2 cubit 10 fingers. Once the volume had been found, the number of man-days required would follow from division by the amount of dirt a worker was supposed to dig out per day (that is, multiplication by its reciprocal), and the barley value from multiplication of the man-days by the daily barley wage of a worker—both again expressed in sexagesimal place-value multiples of basic units for volume, respectively, capacity measure. Once the place-value expression of a metrological value had been found, it would finally have to be reconverted into normal metrology, which would presuppose knowledge of the absolute order of magnitude to which the place-value numbers corresponded.

The conversion of the metrological units into place-value units and vice versa was made by means of "metrological tables". These would tell not only the conversion of the single units but also their multiples. For instance, the table for horizontal extension would start as shown in Fig. 3,[9] stating not only that a finger is 10 (namely 10´´ NINDAN) but also that two fingers are 20, etc. These tables were copied so oft in school that future calculators knew them by heart; in this way, conversion of a composite expression like 8 NINDAN 3 cubit was reduced to an addition—there was no need to multiply 5 (the converted value of the cubit) by 3.

Such metrological tables existed for weight, capacity, horizontal and vertical extension and area (volumes were measured in area units, the standard area 1 SAR = 1 NINDAN$^2$ being presupposed to be provided with a default thickness of 1 cubit).

A final group of tables contains *technical constants*.[10] Some of these are norms for work—how much dirt is a worker supposed to dig out in a day or to carry a fixed distance in a day, etc. Others might serve in geometrical computation. For the circle area, we find the constant 5—to be understood as 5´: Under the assumption that the perimeter $p$ is 3 times the diameter $d$, the area is indeed $\frac{1}{12}p^2 = 5' \times p^2$. For the diameter, we find the constant 20 (to be understood as 20´): $d = \frac{1}{3}p = 20' \times p$.

Technical constants that might turn up as divisors were chosen as regular numbers, preferably as numbers appearing in the table of reciprocals; that explains

[8]Obviously, any unit $60^n$ NINDAN would do in principle, but since the NINDAN was an existing unit abundantly used in practical life, we may take for granted that the calculators would think in terms of NINDAN and not, for instance, $\frac{1}{60}$ NINDAN.

[9]Translated from the edition in Proust (2008: 42). The actual specimen goes no further, but it is only the beginning of the ideal complete table, known in total from the combination of such fragments.

[10]These are less well-treated in the general literature than the arithmetical tables. A recent thorough analysis is Robson (1999).

| | | | |
|---|---|---|---|
| 1 finger | 10 | 1/3 cubit | 1.40 |
| 2 fingers | 20 | 1/2 cubit | 2.30 |
| 3 fingers | 30 | 2/3 cubit | 3.20 |
| 4 fingers | 40 | 1 cubit | 5 |
| 5 fingers | 50 | 1 1/3 cubit | 6.40 |
| 6 fingers | 1 | 1 1/2 cubit | 7.30 |
| 7 fingers | 1.10 | 1 2/3 cubit | 8.20 |
| 8 fingers | 1.20 | 2 cubits | 10 |
| 9 fingers | 1.30 | … | |

**Fig. 3** Beginning of the metrological table for horizontal extension

why the reciprocals of such numbers would also turn up as principal numbers for multiplication.

Exactly as the floating-point calculations on the slide rule of an engineer fifty years ago, calculations in the place-value system could only serve for intermediate calculations, and they would normally leave just as few traces in the written record. We have the various tables and even evidence for the way they constituted an ordered curriculum during the Old Babylonian period (2000–1600 BCE)—remains from Ur III are very rare, just sufficient to show that the system had been created. We also have Old Babylonian student exercises of multiplication showing two factors and their product (but no intermediate calculations). However, the above description of the full combined use of the various tables is based on reconstruction and on Old Babylonian mathematical school texts,[11] not on real administrative records.

## Addition and Abacus

As mentioned, we have no indication that the place-value notation was of any use for additions and subtractions. In particular, we have no exercise tablets with additions as we have for multiplications—if the multiplication goes beyond what follows directly from the multiplication tables and asks for the addition of partial products, these leave no traces on the tablet and must therefore have been manipulated in a different medium. For instance, the outcome of 1.03.45 × 1.03.45 is stated directly to be 1.07.44.03.45 (UET 6/2 222, in Robson (1999: 252))—certainly a calculation few if anybody would be able to perform by mere mental calculation combining multiplication table entries.

"… leave no traces"—or rather, leave only rare indirect traces (Høyrup 2002b). One of these is problem #12 of the text BM 13901 (ed. Neugebauer 1935: III, 3), where the outcome of the multiplication $10'50'' \times 10'50''$ is stated to be $1'57''\underline{46}'''40''''$ instead of $1'57''\underline{21}'''40''''$ (since the problem is inhomogeneous of the second degree, we can see which absolute order of magnitude is intended).

[11]See, for instance, VAT 8389 #1, as discussed in Høyrup (2002a: 77–82).

25 (25′′′) has thus been added erroneously in the calculation to 21 (21′′′), and the only reasonable explanation for that is that 25 has arisen as a partial product and has been inserted twice instead of once—something that could never happen in our paper algorithm, where we see which steps have already been performed.

How could 25 arise as a partial product in the right order of magnitude? There seems to be only one straightforward way, namely by a calculation of 50′′ × 50′′ as (5 × 5) (10′′ × 10′′) = 25 × 1′′′40′′′′ = 25′′′ + 25 × 40′′′′ = 25′′′ + 16′′′40′′′′.

That may sound strange. We know multiplication tables with 50 as principal number and thus containing 50 × 50. However, if we compare the number of extant tables of this kind with the number of surviving copies of the table of reciprocals we see that it was hardly learned by heart—learning tables by heart was done by repeated copying. [12] So, the conclusion appears to be that at least this computation was made on an instrument where you had to remember where you were in the process because a step, once performed, became invisible—similarly to modern pocket calculators. The instrument could be some kind of reckoning board making use of counters, but it is difficult to exclude other possibilities (however, the numbers occasionally inscribed in empty spaces of mathematical tablets offer good evidence that writing on clay with subsequent deletion, in the style of a medieval dust abacus, was not the medium—it is also difficult to see why the number of "places" available on a support of this kind should be restricted). In any case, subtraction was spoken of (in different Sumerian words, and thus without *linguistic* continuity) during Ur III and in Seleucid times as "lifting up", which can hardly refer to anything but the removal of counters.

The text TMS XIX #2 (ed. Bruins and Rutten 1961: 103, pl. 29) provides us with supplementary evidence. Here, two errors are made.[13] In line 4, 14′48′′53′′′20′′′′ × 14′48′′53′′′20′′′′ is stated to be 3′39′′[28′′′]$\underline{44''''26^{(5)}}$40$^{(6)}$, not 3′39′′28′′′$\underline{43''''27^{(5)}}$24$^{(6)}$26$^{(7)}$40$^{(8)}$). In lines 6–7, 11′′6′′′40′′′′ is added to the number 3′39′′[28′′′]44′′′′26$^{(5)}$40$^{(6)}$, and the result is stated to be 3′50′′$\underline{36'''43''''}$40$^{(5)}$ instead of 3′50′′$\underline{35'''24''''26^{(5)}}$40$^{(6)}$. In the former case, a string "43 27" has been changed into "44 26", after which the repeated "4 26" causes the calculator to change "44 26 24 26 40" into "44 26 40" (the number is used further on and therefore cannot be a copyist's mistake). The second error is more complex, but even here it looks as if a unit has been misplaced in the order of fourths instead of that of thirds (see Høyrup 2002b: 196).

All in all, it thus seems that numbers were represented by counters placed on a counting board (in cases or whatever was used to keep together counters belonging to the same group) in such a way that a unit in one order of magnitude could easily

[12]Neugebauer and Sachs (1945: 12, 20) lists 14 standard tables of reciprocals but only one "single multiplication table" (the type that reflects training) with principal number 50. In Neugebauer (1935: I, 10–13, 36), the numbers are, respectively, 25 and 0.

[13]In both cases, Evert Bruins's transliteration differs from Marguerite Rutten's hand copy of the cuneiform, but since the tablet is one of those which the Louvre had mislaid, the transliteration is based on the hand copy and not on fresh collation with the tablet; the deviations must hence be due to erroneous readings or to misguided attempts to repair. I therefore build on the hand copy.

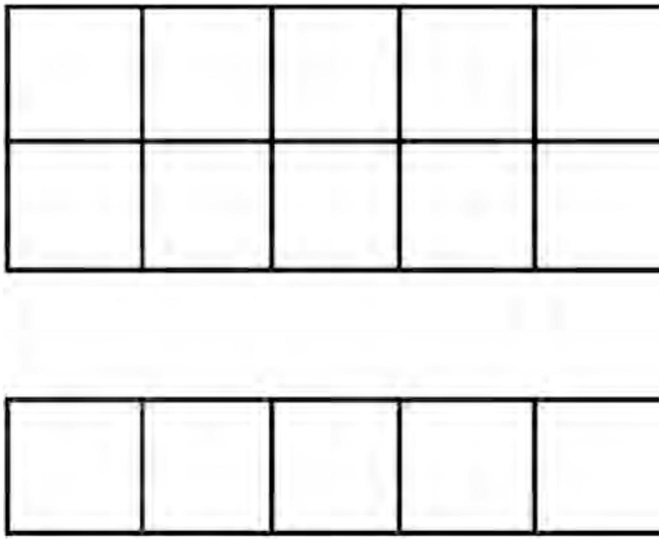

**Fig. 4** Two possible configurations of the Old Babylonian abacus

be misplaced or pushed accidentally into a neighbouring order but not as easily into the tens of the same or a neighbouring order of magnitude. One possibility—probably the most obvious *for us*—is shown in the upper part of Fig. 4, but the configuration shown in the lower part is also possible, provided distinct counters were used for ones and tens (which would correspond to the written numerals).

This concerns how additive (and, we may assume, subtractive) operations were performed in the context of place value computation during the Old Babylonian period. As Proust, Christine (2000) has discovered, however, other error types offer further insight. In a large table of many-digit reciprocals from the Seleucid epoch (AO 6456), repeatedly two sexagesimal places are added together, 45.7 becoming 52, 40.14 appearing as 54, etc. This happens only in the interior of numbers of more than 4 digits, as if computations had been performed on an instrument of limited capacity. An Old Babylonian table listing continued doublings of 2.5 (N3958; 2.5 until $2.5 \times 2^{39}$) gives support to that interpretation: when the numbers grow beyond 5 places, they are written as two numbers separated by a separation character, apparently corresponding to calculations on two separate devices (in the end, when space in the column becomes scarce, the separation character is omitted). The first such number "10 + 6.48.53.20" can be reduced without difficulty to 10.6.48.53.20; soon, however, the right-hand part itself grows beyond five places, and the correct interpretation of "5.20 + 3.38.4.26.40" would be 5.23.38.04.26.40, but performing this operation correctly asks for meticulous book-keeping about places, and errors as those that abound in the Seleucid table are easily explained—as formulated by Proust (2000: 302), they are the "scars of recombination of two separate pieces" (or even, in one entry, three pieces).

Proust also suggests that the device may have carried the name "the hand", pointing to the term ŠU NU TAGA, "which the hand cannot grasp" used about $60^4$, a five-place number, and referring to the present author for supplementary evidence. ŠU NU TAGA is known from Old Babylonian times, but the supplementary evidence spans most of Mesopotamian history: already in the mid-third millennium, ŠU.NIGIN, "the hand holds", designates the total of an account (below, we shall encounter more evidence from the same epoch); in the Old Babylonian text $Db_2$-146, an intermediate result is put "on your hand" and referred to afterwards as "your hand" (Høyrup 2002a: 258*f*), and the astronomical procedure text BM 42282 + 42294 (probably between the sixth and the fourth century BCE (ed. Brack-Bernsen and Hunger 2008)) prescribes that "you hold in your hand your year", which makes no sense unless the current year is inserted in a calculational procedure or device.

In texts from Ur III as well as the Seleucid epoch, we also see that subtraction is spoken of as "taking up" or "lifting up" (ZI in Ur III, NIM in Seleucid times). The introduction of a new word in late times probably means that the terms describe an extra-linguistic operation—presumably to "take up" counters from the counting board.

The early appearance of the name "hand" shows that this board is much older than the place-value system. However, the system for counting had been sexagesimal since the appearance of writing in the later fourth millennium BCE, only with distinct signs for 1, 10, 60, 600, 3600, 36,000 and 216,000; it was thus an absolute-value system. A counting board that would serve calculation in this system would therefore not only be of equal use for addition and subtraction of place-value numbers, it may even have inspired the very invention of the place-value notation—which however was to remain "in the air" until the full *system* with appurtenant tables was devised.

## The Centesimal System and the Decadic Notation

An interesting further or parallel development was discovered a few years ago in Old Babylonian Mari and other cities in the Middle-Euphrates region, towards the Mesopotamian north-west: a place-value system with base 100 (Chambon 2012). It was only used for integers and served the counting of people and quantities of the capacity unit SILÀ. It uses the same basic signs for 1 and 10 as the sexagesimal place-value system, and it is therefore likely to be an adaption of the already known sexagesimal precursor;[14] however, a partially similar system is found in Ebla somewhat to the west around 2400 BCE, yet indicating the order of magnitude of places by mean of number words for 100, 1000 and 10,000 and using the signs of the Sumerian absolute-value counting within each place. The Ebla notation may

[14]Some Mari scribes were trained in sexagesimal place-value arithmetic in the early eighteenth century, so it was not unfamiliar. Moreover, the sign for units was differently oriented in the place value and in the traditional absolute-value systems (vertical respectively horizontal), and the centesimal notation agrees on this account with the sexagesimal place-value system.

also have been inspired by an abacus (of a type corresponding to the local spoken numeral system, which was decadic, that is, with base 10), but it can just as well have been a direct emulation of the way a number like 36,892 was spoken. Similarly, the later Mari system may have received inspiration from the Ebla notation and not only from the sexagesimal place-value system, or the similarity may be accidental, caused by the shared decadic spoken numerals.

Whereas the Sumerian spoken numeral system had been sexagesimal, the numerals of Semitic languages are indeed decadic, as are those of Indoeuropean languages. That is the underlying reason that Ebla, Mari and other north-western cities made use of the centennial system—the native language in this region was Amorite or Akkadian[15] or some other Semitic dialect. In the former Sumerian south, it is likely that the daily language was already Akkadian during Ur III, even though the official language of the state (and thus the language of scribehood) was still Sumerian; it certainly was when the Old Babylonian mathematical texts were written, but outside the area just mentioned the impact of general language on number writing was more modest: 5782 would be written 5 *līm* 7 *mē* 82 (5 thousand 7 hundred 82, 82 being written in the traditional Sumerian absolute-value system, as 60 + 20 + 2), as it had been in Ebla.

We have no—and in all probability there *were* no—conversion tables between this almost-decadic number notation and the sexagesimal notation. In cases where the numbers did not enter calculations that would have no importance; however, if they were to be added (which might happen if, for example, they counted numbers of workers engaged in various parts of a larger project), we may speculate whether the counting board was used for this purpose with a different understanding of its structure; we may also guess that such non-standard ways to use the abacus might have served for operation on for instance capacity measures,[16] in particular before the implementation of the place-value system—but both suggestions remain mere conjectures.

## Third Millennium Difficult Division

As mentioned already, we have evidence that Old Babylonian calculators were able to find approximate reciprocals of irregular numbers. However, we have no hints as to the methods that were used.

[15]Akkadian is the language whose main dialects in the second and first millennium are Babylonian and Assyrian.

[16]The fundamental capacity unit was a SILÀ (ca. 1 l). In Ur III and the Old Babylonian period, it was subdivided sexagesimally into 60 GÍN and the GÍN again in 180 ŠE. 10 SILÀ were 1 BÁN, and 6 BÁN constituted 1 BARIGA. 5 BARIGA, finally, made up a GUR, and GUR were counted in absolute-value sexagesimal numbers. So, for calculating grain quantities (where SILÀ would normally be the smallest unit taken into account), columns or cases with values 1, 10, 60 and 300 for successive cases or columns would be adequate.

From the mid-third millennium BCE, on the other hand, we have three texts that show something about how large round numbers could be divided by irregular divisors.

Two of the texts are from the city Šuruppak and can be dated to c. 2550 BCE (Høyrup 1982). They both deal with the distribution of a "storehouse" of barley to workers, each of whom receives 7 SILÀ. The "storehouse" of Šuruppak of the time was expected to contain 40` (=2400) GUR.MA, each GUR.MA ("great GUR") consisting of 8`(=480) SILÀ (that is, 1 "storehouse" = 1,152,000 SILÀ). The problem is thus to divide 2400 × 480 by 7. One of the texts (TMS 50) gives the correct answer 45``42`51 (=164,571) men, 3 SILÀ being "left on the hand", that is, left as remainder on the counting board. The other (TMS 671), however, finds 45``36` (=164,160) men. As it turns out, this is an intermediate result if the correct solution is found in the following way: first, we divide the number of GUR.MA in a storehouse by 7; that is, we find how many times 7 GUR.MA is contained in 40` GUR.MA); the answer is 342 times, with a remainder of 6 GUR.MA. Then, we multiply by the number of times 7 SILÀ is contained in 7 GUR.MA, which is obviously 8` = 480 times, getting 164,160—the very result obtained in the second text. This is thus as many men as will get 7 SILÀ each from the storehouse *if we forget about the remainder*. However, if we divide the remainder of 6 GUR.MA by 7 SILÀ, we find that 411 more men will receive their ration (in total thus 164,571 men, the result stated in the first text), with a remainder of 3 SILÀ.

It is impossible to find reasonable alternative procedures that also have the result stated in the mistaken text as an intermediate result. We may therefore be confident that this was how the result was reached; the analysis leaves open the question, however, how 40` (=2400) and 8`(=480) were divided by 7.

The third text (TM.75.G.1392) is from Ebla and from c. 2400 BCE; I follow Jöran Friberg's interpretation (1986: 16–21). The text appears to show a method for finding out how much grain has to be distributed to 260,000 persons, if 33 persons receive 1 *gú-bar*.[17]

It is stated (for simplicity, the sub-units are translated as fractions in the left column, while the middle column reduces these fractions; both follow Friberg) that

| | | |
|---|---|---|
| $3\frac{4}{120}$ *gú-bar* | ($=3\frac{1}{30}$ *gú-bar*) | for 100 persons |
| $30\frac{6}{20}$ *gú-bar* | ($=30\frac{3}{10}$ *gú-bar*) | for 1000 persons |
| $303\frac{4}{120}$ *gú-bar* | ($=303\frac{1}{30}$ *gú-bar*) | for 10,000 persons |
| $3030\frac{6}{20}$ *gú-bar* | ($=3030\frac{3}{10}$ *gú-bar*) | for 100,000 persons |
| $6060\frac{1}{2}\frac{2}{20}\frac{2}{120}$ *gú-bar* | ($=6060\frac{6}{10}\frac{1}{60}$ *gú-bar*) | for 200,000 persons |
| $1818\frac{24}{120}$ *gú-bar* | ($=1818\frac{2}{10}$ *gú-bar*) | for 60,000 persons |

In all: 7879 *gú-bar* of barley for 260,000 persons.

[17]The *gú-bar* is a local Ebla unit; the transliteration is written in italics because it renders a syllabic writing of a Semitic word.

Since 33 persons receive 1 *gú-bar*, 3 × 33 = 99 persons receive 3 *gú-bar*. 100 = 99 + 1 persons therefore should receive 3 *gú-bar* + $\frac{1}{33}$ *gú-bar*, etc. Firstly we notice, however, that all values are slightly rounded: $\frac{1}{33}$ is replaced by $\frac{1}{30}$ and $\frac{10}{33} = \frac{30}{99}$ by $\frac{3}{10}$; in the final summation, $\frac{1}{2}\frac{2}{20}\frac{2}{120} + \frac{4}{20} = \frac{49}{60}$ is approximated as 1. Secondly, we observe that the successive values are not obtained by simple multiplication (by 10 respectively 2). Precisely how the values in the successive lines are found we cannot decide, but in any case we see that the division of 260,000 by 33 (or, in classical formulation, the measurement of 260,000 persons by 33 persons) is found through filling-out: first, by decupling and doubling, we go as far as possible, that is, until 200,000 persons; 60,000 persons remain, whose allocation is probably found by multiplying the allocation of 10,000 person by 6 (no rounding needed). Quite plausibly, the simpler divisions in the single lines were carried out in a similar way.

We further observe that the trick used in the Šuruppak texts is different from the method of the Ebla text (while, of course, its simpler divisions may or may not have been performed as fillings). The two texts do not present us with a standard way (and certainly not with an "algorithm") for performing divisions by irregular numbers; instead, they represent systematic exploration—in Friberg's words (1986: 22),

> the "current fashion" among mathematicians about four and a half millennia years ago was to study non-trivial division problems involving large (decimal or sexagesimal) numbers and "non-regular" divisors such as 7 and 33.

Nothing prevents, however, that such exploration could eventually lead to the creation of standard methods and that these would come to be used by the Old Babylonian calculators.

## Long-Time Developments—Summary and Conclusion

Through accounting and metrologies, Mesopotamian mathematics can be followed back to the "proto-literate" period (c. 3300–3000 BCE) where writing was created (created, indeed, in order to serve in accounting, by providing the context that gave meaning to the numbers of the accounts). But we know nothing about the computational techniques in use by then.

Only Šuruppak, around 2550 BCE, provides us with some insights. Šuruppak presents us with evidence of several kinds that the "hand" reckoning board was in use, and it gives us the first example of the division by an irregular number. From Šuruppak, we also have the earliest table of squares, where the side is given in length metrology and the area measured in area units (Neugebauer 1935: I, 91).

Three more square tables come from the following century (Edzard 1969; Feliu 2012; Friberg 2007: 419–427); one of them also lists rectangular areas, one of the sides being constantly 1` NINDAN.

During the centuries preceding Ur III, we find several instances of notations that suggest ongoing groping for the place-value idea, but almost all contain mistakes showing that the *system* was not yet in existence (Powell 1976).[18] The system was only to be created during Ur III—and its complex combination of a number notation and the variety of table types without which it would be of no use shows that it was certainly a deliberate creation, not the outcome of accumulated accidental developments.

The Ur III state broke down around 2000 BCE, but the scribes of the less centralized Old Babylonian successor states were still trained in place-value calculation. After the collapse around 1600 of the final Old Babylonian state, the Babylon of the Hammurabi dynasty, we know less. Scholar-scribes were still taught some rudiments—Ashurbanipal, the last important Assyrian king (r. 668–631 BCE), who had originally been meant to become a high priest, boasts that he is able to perform multiplications and find reciprocals. That seems to be the high point of the mathematics he knows about: in same text, he claims to be able to read tablets "from before the flood", that is, from the mid-third millennium, which appears not to be true—but real scholar-scribes at his court could do it. Those who took care of mathematical administration after the collapse of the Old Babylonian state were hardly scholar-scribes—there is evidence that only the most basic vocabulary surrounding the place-value system was conserved in Sumerian. However, at the creation of mathematical astronomy from the seventh century BCE onward, the place-value system again came in use albeit within a very restricted environment. As we have seen, this environment still used the "hand" reckoning board, and it also knew the trailing part algorithm.

Mathematical astronomy survived at least until the late first century CE (Hunger and de Jong 2014); by then, mathematical administration had given up the cuneiform heritage since long. The disappearance of mathematical astronomy therefore entailed the final demise of the Mesopotamian calculation techniques, after their having been practised for more than 2000, some of them for at least 2500, perhaps 3400 years.

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[18]Whiting (1984) goes further than Powell in his claims, but his argument suffers from a lack of distinction between *sexagesimalization* and place value.

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